# Adaptive Control for a Class of Nonlinear Systems with a Time-Varying Structure

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**Abstract**

In this paper we present a direct adaptive control method for a class of uncertain nonlinear systems with a time-varying structure. We view the nonlinear systems as composed of a finite number of "pieces," which are interpolated by functions that depend on a possibly exogenous scheduling variable. We assume that each piece is in strict feedback form, and show that the method yields stability of all signals in the closed-loop, as well as convergence of the state vector to a residual set around the equilibrium, whose size can be set by the choice of several design parameters. The class of systems considered here is a generalization of the class of strict feedback systems traditionally considered in the backstepping literature. We also provide design guidelines based on $\mathcal{L}_\infty$ bounds on the transient.

## 1 Introduction

The field of nonlinear adaptive control developed rapidly in the last decade. The paper [1] and others gave birth to an important branch of adaptive control theory, the nonlinear on-line function

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approximation based control, which includes neural (e.g., in [2]) and fuzzy (e.g., in [3]) approaches (note that there are several other relevant works on neural and fuzzy control, many of them cited in the references within the above papers). The neural and fuzzy approaches are most of the time equivalent, differing between each other only for the structure of the approximator chosen [4]. Most of the papers deal with indirect adaptive control, trying first to identify the dynamics of the systems and eventually generating a control input according to the certainty equivalence principle (with some modification to add robustness to the control law), whereas very few authors (e.g., in [4, 5]) use the direct approach, in which the controller *directly* generates the control input to guarantee stability.

Plants whose dynamics can be expressed in the so called "strict feedback form" have been considered, and techniques like backstepping and adaptive backstepping [6] have emerged for their control. The papers [2, 7] present an extension of the tuning functions approach in which the nonlinearities of the strict feedback system are not assumed to be parametric uncertainties, but rather completely unknown nonlinearities to be approximated on-line with nonlinearly parameterized function approximators. Both the adaptive methods in [6] and in [2, 7] attempt to approximate the dynamics of the plant on-line, so they may be classified as indirect adaptive schemes.

In this paper, we have combined an extension of the class of strict feedback systems considered in [2, 7] with the concept of a dynamic structure that depends on time, so as to propose a class of nonlinear systems with a time-varying structure, for which we develop a *direct* adaptive control approach. This class of systems is a generalization of the class of strict feedback systems traditionally considered in the literature. Moreover, the direct adaptive control developed here is, to our knowledge, the first of its kind in this context, and it presents several advantages with respect to indirect adaptive methods, including the fact that it needs less plant information to be implemented.

## 2 Direct Adaptive Control

Consider the class of continuous time nonlinear systems given by

$$\begin{aligned} \dot{x}_i &= \sum_{j=1}^{R} \rho_j(v) \left(\phi_i^j(X_i) + \psi_i^j(X_i) x_{i+1}\right) \\ \dot{x}_n &= \sum_{j=1}^{R} \rho_j(v) \left(\phi_n^j(X_n) + \psi_n^j(X_n) u\right) \end{aligned} \tag{1}$$

where $i = 1, 2, \ldots, n-1$, $X_i = [x_1, \ldots, x_i]^\top$, and $X_n \in \mathbb{R}^n$ is the state vector, which we assume measurable, and $u \in \mathbb{R}$ is the control input. The variable $v \in \mathbb{R}^q$ may be an additional input or a possibly exogenous "scheduling variable." We assume that $v$ and its derivatives up to and including the $(n-1)^{th}$ one are bounded and available for measurement, which may imply that $v$ is given by an external dynamical system. The functions $\rho_j$, $j = 1, \ldots, R$ may be considered to be "interpolating functions" that produce the time-varying structural nature of system (1), since they combine $R$ systems in strict feedback form (given by the $\phi_i^j$ and $\psi_i^j$ functions, $i = 1, \ldots, n$, $j = 1 \ldots, R$) and the combination depends on time through the variable $v$ (thereby, the dynamics of the plant may be different at each time point depending on the scheduling variable). Here, we assume that the functions $\rho_j$ are $n$ times continuously differentiable, and that they satisfy, for all $v \in \mathbb{R}^q$, $\sum_{j=1}^{R} \rho_j(v) < \infty$ and $\left|\frac{\partial^i \rho_j(v)}{\partial v^i}\right| < \infty$. Denote for convenience $\phi_i^c(X_i, v) = \sum_{j=1}^{R} \rho_j(v) \phi_i^j(X_i)$ and $\psi_i^c(X_i, v) = \sum_{j=1}^{R} \rho_j(v) \psi_i^j(X_i)$. We will assume that $\phi_i^c$ and $\psi_i^c$ are sufficiently smooth in their arguments, and that they satisfy, for all $X_i \in \mathbb{R}^i$ and $v \in \mathbb{R}^q$, $i = 1, \ldots, n$, $\phi_i^c(0, v) = 0$ and $\psi_i^c(X_i, v) \neq 0$.

Here, we will develop a direct adaptive control method for the class of systems (1). We assume that the interpolation functions $\rho_j$ are known, but the functions $\phi_i^j$ and $\psi_i^j$ (which constitute the underlying time-varying dynamics of the system) are unknown. In an indirect adaptive methodology one would attempt to identify the unknown functions and then construct a stabilizing control law based on the approximations to the plant dynamics. Here, however, we will postulate the existence of an ideal control law (based on the assumption that the plant belongs to the class of systems

(1)) which possesses some desired stabilizing properties, and then we devise adaptation laws that attempt to approximate the ideal control equation. This approximation will be performed within a compact set $\mathcal{S}_{x_n} \subset \mathbb{R}^n$ of arbitrary size which contains the origin. In this manner, the results obtained are semi-global, in the sense that they are valid as long as the state remains within $\mathcal{S}_{x_n}$, but this set can be made as large as desired by the designer. In particular, with enough plant information it can be made large enough that the state never exits it, since, as will be shown a bound can be placed on the state transient. Furthermore, as will be indicated below, the stability can be made global by using bounding control terms.

For each vector $X_i$ we will assume the existence of a compact set $\mathcal{S}_{x_i} \subset \mathbb{R}^i$ specified by the designer. We will consider trajectories within the compact sets $\mathcal{S}_{x_i}$, $i = 1, \ldots, n$, where the sets are constructed such that $\mathcal{S}_{x_i} \subset \mathcal{S}_{x_{i+1}}$, for $i = 1, \ldots, n-1$. We assume the existence of bounds $\underline{\psi}_i^c$, $\bar{\psi}_i^c \in \mathbb{R}$, and $\psi_{i_d}^c \in \mathbb{R}$, $i = 1, \ldots, n$ (*not necessarily known*), such that for all $v \in \mathbb{R}^q$ and $X_i \in \mathcal{S}_{x_i}$, $i = 1, \ldots, n$,

$$\begin{aligned} &0 < \underline{\psi}_i^c \leq \psi_i^c(X_i, v) \leq \bar{\psi}_i^c < \infty \\ &\left|\dot{\psi}_i^c\right| = \left|\sum_{j=1}^{R} \left( \frac{\partial \rho_j(v)}{\partial v} \dot{v} \psi_i^j(X_i) + \rho_j(v) \frac{\partial \psi_i^j(X_i)}{\partial X_i} \dot{X}_i \right)\right| \leq \psi_{i_d}^c. \end{aligned} \tag{2}$$

This assumption implies that the affine terms in the plant dynamics have a bounded gain and a bounded rate of change. Since the functions $\psi_i^c$ are assumed continuous, they are therefore bounded within $\mathcal{S}_{x_i}$. Similarly, note that even though the term $|\dot{X}_i|$ may not necessarily be globally bounded, it will have a constant bound within $\mathcal{S}_{x_i}$ due to the continuity assumptions we make. Therefore, assumption (2) will always be satisfied within $\mathcal{S}_{x_n}$. Moreover, in the simplest of cases, the first part of assumption (2) is satisfied globally when the functions $\psi_i^j$ are constant or sector bounded for all $X_i \in \mathbb{R}^i$.

The class of plants (1) is, to our knowledge, the most general class of systems considered so far within the context of adaptive control based on backstepping. In particular, in both [6] and [2, 7], which are indirect adaptive approaches, the input functions $\psi_i^j$ are assumed to be constant

for $i = 1, \ldots, n$. This assumption allows the authors of those works to perform a simpler stability analysis, which becomes more complex in the general case [8]. Also, the addition of the interpolation functions $\rho_j$, $j = 1, \ldots, R$, extends the class of strict feedback systems to one including systems with a time-varying structure [9], as well as systems falling in the domain of gain scheduling (where the plant dynamics are identified at different operating points and then interpolated between using a scheduling variable). Note that if we let $R = 1$ and $\rho_1(v) = 1$ for all $v$, together with $\psi_i^c = 1$, $i = 1, \ldots, n$, we have the particular case considered in [2, 7].

The direct approach presented here has several advantages with respect to indirect approaches such as in [6, 2, 7]. In particular, bounds on the input functions $\psi_i^j$ are only assumed to exist, but need *neither to be known nor to be estimated*. This is because the ideal law is formulated so that there is not an explicit need to include information about the bounds in the actual control law. Moreover, although assumption (2) appears to be more restrictive than what is needed in the indirect adaptive case, it is in fact not so due to the fact that the stability results are semi-global (i.e., since we are operating within the compact sets $\mathcal{S}_{x_n}$, continuity of the affine terms automatically implies the satisfaction of the second part of assumption (2)).

## 2.1 Direct Adaptive Control Theorem

Next, we state our main result and then show its proof[1]. For convenience, we use the notation $\nu_i = [v, \dot{v}, \ldots, v^{(i-1)}] \in \mathbb{R}^{q \times i}$, $i = 1, \ldots, n$.

**Theorem 1**: *Consider system* (1) *with the state vector $X_n$ measurable and the scheduling matrix $\nu_{n-1}$ measurable and bounded, together with the above stated assumptions on $\phi_i^c$, $\psi_i^c$ and $\rho_j$, and* (2). *Assume also that $\nu_i(0) \in \mathcal{S}_{v_i} \subset \mathbb{R}^{q \times i}$, $X_i(0) \in \mathcal{S}_{x_i} \subset \mathbb{R}^i$, $i = 1, \ldots, n$, where $\mathcal{S}_{v_i}$ and $\mathcal{S}_{x_i}$ are compact sets specified by the designer, and large enough that $\nu_i$ and $X_i$ do not exit them. Consider the diffeomorphism $z_1 = x_1$, $z_i = x_i - \hat{\alpha}_{i-1} - \alpha_{i-1}^s$, $i = 2, \ldots, n$, with $\hat{\alpha}_i(X_i, \nu_i) = \sum_{j=1}^R \rho_j(v) \hat{\theta}_{\alpha_i^j}^\top \zeta_{\alpha_i^j}(X_i, \nu_i)$ and $\alpha_i^s(z_i, z_{i-1}) = -k_i z_i - z_{i-1}$, with $k_i > 0$ and $z_0 = 0$. Assume the functions $\zeta_{\alpha_i^j}(X_i, \nu_i)$ to be at*

[1] We will generally omit the arguments of functions for brevity.

*least $n-i$ times continuously differentiable, and to satisfy, for $i = 1, \ldots, n$, $j = 1, \ldots, R$,*

$$\left| \frac{\partial^{n-i} \zeta_{\alpha_i^j}}{\partial [X_i, \nu_i]^{n-i}} \right| < \infty. \tag{3}$$

*Consider the adaptation laws for the parameter vectors $\hat{\theta}_{\alpha_i^j} \in \mathbb{R}^{N_{\alpha_i^j}}$, $N_{\alpha_i^j} \in \mathbb{N}$, $\dot{\hat{\theta}}_{\alpha_i^j} = -\rho_j \gamma_{\alpha_i^j} \zeta_{\alpha_i^j} z_i - \sigma_{\alpha_i^j} \hat{\theta}_{\alpha_i^j}$, where $\gamma_{\alpha_i^j} > 0$, $\sigma_{\alpha_i^j} > 0$, $i = 1, \ldots, n$, $j = 1, \ldots, R$ are design parameters. Then, the control law $u = \hat{\alpha}_n + \alpha_n^s$ guarantees boundedness of all signals and convergence of the states to the residual set*

$$\mathcal{D}_d = \left\{ X_n \in \Re^n : \sum_{i=1}^{n} z_i^2 \leq \frac{2\underline{\psi}_m W_d}{\beta_d} \right\}. \tag{4}$$

*where $\underline{\psi}_m = \min_{1 \leq i \leq n} \bar{\psi}_i^c$, $\beta_d$ is a constant, and $W_d$ measures approximation errors and ideal parameter sizes, and its magnitude can be reduced through the choice of the design constants $k_i$, $\gamma_{\alpha_i^j}$ and $\sigma_{\alpha_i^j}$.*

*Proof:* The proof requires $n$ steps, and is performed inductively. First, let $z_1 = x_1$, and $z_2 = x_2 - \hat{\alpha}_1 - \alpha_1^s$, where $\hat{\alpha}_1$ is the approximation to an ideal signal $\alpha_1^*$ ("ideal" in the sense that if we had $\hat{\alpha}_1 = \alpha_1^*$ we would have a globally asymptotically stable closed loop without need for the stabilizing term $\alpha_1^s$), and $\alpha_1^s$ will be given below. Let $c_1 > 0$ be a constant such that $c_1 > \frac{\psi_{1_d}^c}{2\underline{\psi}_1^c}$, and $\alpha_1^*(x_1, v) = \frac{1}{\psi_1^c} \left( -\phi_1^c - c_1 z_1 \right)$. Since the ideal control $\alpha_1^*$ is smooth, it may be approximated with arbitrary accuracy for $v$ and $x_1$ within the compact sets $\mathcal{S}_{v_1} \subset \mathbb{R}^q$ and $\mathcal{S}_{x_1} \subset \mathbb{R}$, respectively, as long as the size of the approximator can be made arbitrarily large.

For approximators of finite size let $\alpha_1^*(x_1, v) = \sum_{j=1}^{R} \rho_j(v) \theta_{\alpha_1^j}^{*\top} \zeta_{\alpha_1^j}(v, x_1) + \delta_{\alpha_1}(v, x_1)$, where the parameter vectors $\theta_{\alpha_1^j}^* \in \mathbb{R}^{N_{\alpha_1^j}}$, $N_{\alpha_1^j} \in \mathbb{N}$, are optimum in the sense that they minimize the representation error $\delta_{\alpha_1}$ over the set $\mathcal{S}_{x_1} \times \mathcal{S}_{v_1}$ and suitable compact parameter spaces $\Omega_{\alpha_1^j}$, and $\zeta_{\alpha_1^j}(x_1, v)$ are defined via the choice of the approximator structure (see [10] for an example of a choice for $\zeta_{\alpha_i^j}$). The parameter sets $\Omega_{\alpha_1^j}$ are simply mathematical artifacts. As a result of the stability proof the approximator parameters are bounded using the adaptation laws in Theorem 1, so $\Omega_{\alpha_1^j}$ does not need to be defined explicitly, and no parameter projection (or any other "artificial" means of

keeping the parameters bounded) is required. The representation error $\delta_{\alpha_1}$ arises because the sizes $N_{\alpha_i^j}$ are finite, but it may be made arbitrarily small within $\mathcal{S}_{x_1} \times \mathcal{S}_{v_1}$ by increasing $N_{\alpha_i^j}$ (i.e., we assume the chosen approximator structures possess the "universal approximation property"). In this way, there exists a constant bound $d_{\alpha_1} > 0$ such that $|\delta_{\alpha_1}| \leq d_{\alpha_1} < \infty$. To make the proof logically consistent, however, we need to assume that some knowledge about this bound and a bound on $\theta^*_{\alpha_1^j}$ are available (since in this case it becomes possible to guarantee a priori that $\mathcal{S}_{x_1} \times \mathcal{S}_{v_1}$ is large enough). However, in practice some amount of redesign may be required, since these bounds are typically guessed by the designer

Let $\Phi_{\alpha_1^j} = \hat{\theta}_{\alpha_1^j} - \theta^*_{\alpha_1^j}$ denote the parameter error, and approximate $\alpha_1^*$ with $\hat{\alpha}_1(x_1, v, \hat{\theta}_{\alpha_1^j}; j = 1, \ldots, R) = \sum_{j=1}^R \rho_j(v) \hat{\theta}^\top_{\alpha_1^j} \zeta_{\alpha_1^j}(x_1, v)$. Hence, we have a linear in the parameters approximator with parameter vectors $\hat{\theta}_{\alpha_1^j}$. Note that the structural dependence on time of system (1) is reflected in the controller, because $\hat{\alpha}_1$ can be viewed as using the functions $\rho_j(v)$ to interpolate between "local" controllers of the form $\hat{\theta}^\top_{\alpha_1^j} \zeta_{\alpha_1^j}(x_1, v)$, respectively. Notice that since the functions $\rho_j$ are assumed continuous and $v$ bounded, the signal $\hat{\alpha}_1$ is well defined for all $v \in \mathcal{S}_{v_1}$.

Consider the dynamics of the transformed state, $\dot{z}_1 = \phi_1^c + \psi_1^c(z_2 + \hat{\alpha}_1 + \alpha_1^s) + \psi_1^c(\alpha_1^* - \alpha_1^*) = -c_1 z_1 + \psi_1^c z_2 + \psi_1^c(\hat{\alpha}_1 - \alpha_1^*) + \psi_1^c \alpha_1^s = -c_1 z_1 + \psi_1^c z_2 + \psi_1^c \left( \sum_{j=1}^R \rho_j \Phi^\top_{\alpha_1^j} \zeta_{\alpha_1^j} - \delta_{\alpha_1^j} \right) + \psi_1^c \alpha_1^s$. Let $V_1 = \frac{1}{2\psi_1^c} z_1^2 + \frac{1}{2} \sum_{j=1}^R \frac{\Phi^\top_{\alpha_1^j} \Phi_{\alpha_1^j}}{\gamma_{\alpha_1^j}}$, and examine its derivative, $\dot{V}_1 = \frac{2\psi_1^c(2 z_1 \dot{z}_1) - 2 z_1^2 \dot{\psi}_1^c}{4 {\psi_1^c}^2} + \sum_{j=1}^R \frac{\Phi^\top_{\alpha_1^j} \dot{\Phi}_{\alpha_1^j}}{\gamma_{\alpha_1^j}}$. Using the expression for $\dot{z}_1$, $\dot{V}_1 = -\frac{c_1 z_1^2}{\psi_1^c} + z_1 z_2 + z_1 \sum_{j=1}^R \rho_j \Phi^\top_{\alpha_1^j} \zeta_{\alpha_1^j} - z_1 \delta_{\alpha_1^j} + z_1 \alpha_1^s - \frac{1}{2} z_1^2 \frac{\dot{\psi}_1^c}{{\psi_1^c}^2} + \sum_{j=1}^R \frac{\Phi^\top_{\alpha_1^j} \dot{\Phi}_{\alpha_1^j}}{\gamma_{\alpha_1^j}}$. Choose the adaptation law $\dot{\hat{\theta}}_{\alpha_1^j} = \dot{\Phi}_{\alpha_1^j} = -\rho_j \gamma_{\alpha_1^j} \zeta_{\alpha_1^j} z_1 - \sigma_{\alpha_1^j} \hat{\theta}_{\alpha_1^j}$, with design constants $\gamma_{\alpha_1^j} > 0$, $\sigma_{\alpha_1^j} > 0$, $j = 1, \ldots, R$ (we think of $\sigma_{\alpha_1^j} \hat{\theta}_{\alpha_1^j}$ as a "leakage term"). Also, note that for any constant $k_1 > 0$, $-z_1 \delta_{\alpha_1^j} \leq |z_1| d_{\alpha_1} \leq k_1 z_1^2 + \frac{d^2_{\alpha_1}}{4k_1}$. We pick $\alpha_1^s = -k_1 z_1$.

Notice also that, completing squares, $-\Phi^\top_{\alpha_1^j} \hat{\theta}_{\alpha_1^j} = -\Phi^\top_{\alpha_1^j}(\Phi_{\alpha_1^j} + \theta^*_{\alpha_1^j}) \leq -\frac{|\Phi_{\alpha_1^j}|^2}{2} + \frac{|\theta^*_{\alpha_1^j}|^2}{2}$. Finally, observe that $-\frac{z_1^2}{\psi_1^c} \left( c_1 + \frac{\dot{\psi}_1^c}{2\psi_1^c} \right) \leq -\frac{z_1^2}{\psi_1^c} \left( c_1 - \frac{\psi^c_{1_d}}{2\underline{\psi}_1^c} \right) \leq -\frac{\bar{c}_1 z_1^2}{\psi_1^c}$, with $\bar{c}_1 = c_1 - \frac{\psi^c_{1_d}}{2\underline{\psi}_1^c} > 0$. Then, we obtain $\dot{V}_1 \leq -\frac{\bar{c}_1 z_1^2}{\psi_1^c} - \frac{1}{2} \sum_{j=1}^R \sigma_{\alpha_1^j} \frac{|\Phi_{\alpha_1^j}|^2}{\gamma_{\alpha_1^j}} + z_1 z_2 + \frac{d^2_{\alpha_1}}{4k_1} + \frac{1}{2} \sum_{j=1}^R \sigma_{\alpha_1^j} \frac{\theta^*_{\alpha_1^j}}{\gamma_{\alpha_1^j}}$. This completes the first step of the proof.

We may continue in this manner up to the $n^{th}$ step[2], where we have $z_n = x_n - \hat{\alpha}_{n-1} - \alpha^s_{n-1}$, with $\hat{\alpha}_{n-1}$ and $\alpha^s_{n-1}$ defined as in Theorem 1. Consider the ideal signal $\alpha^*_n(X_n, \nu_n) = \frac{1}{\psi^c_n}\left(\phi^c_n - c_n z_n + \dot{\hat{\alpha}}_{n-1} + \dot{\alpha}^s_{n-1}\right)$ with $c_n > \frac{\psi^c_{n_d}}{2\underline{\psi}^c_n}$. Notice that, even though the terms $\dot{\hat{\theta}}_{\alpha^j_{n-1}}$ appear in $\alpha^*_n$ through the partial derivatives in $\dot{\hat{\alpha}}_{n-1}$, $\hat{\theta}_{\alpha^j_{n-1}}$ does not need to be an input to $\alpha^*_n$, since the resulting product of the partial derivatives and $\dot{\hat{\theta}}_{\alpha^j_{n-1}}$ can be expressed in terms of $z_1, \ldots, z_{n-1}$, $v$ and $\sigma_{\alpha^j_{n-1}} \hat{\alpha}_{n-1}$. To simplify the notation, however, we will omit the dependencies on inputs other than $X_i$ and $\nu_i$, but bearing in mind that, when implementing this method, more inputs may be required to satisfy the proof. Also, note that by assumption (3), $|\alpha^*_n| < \infty$ for bounded arguments. Therefore, we may represent $\alpha^*_n$ with $\alpha^*_n(X_n, \nu_n) = \sum_{j=1}^R \rho_j(v) \theta^{*\top}_{\alpha^j_n} \zeta_{\alpha^j_n}(X_n, \nu_n) + \delta_{\alpha_n}(X_n, \nu_n)$ for $X_n \in \mathcal{S}_{x_n} \subset \mathbb{R}^n$ and $\nu_n \in \mathcal{S}_{v_n} \subset \mathbb{R}^{q \times n}$. The parameter vector $\theta^*_{\alpha^j_n} \in \mathbb{R}^{N_{\alpha^j_n}}$, $N_{\alpha^j_n} \in \mathbb{N}$ is an optimum within a compact parameter set $\Omega_{\alpha_n}$, in a sense similar to $\theta^*_{\alpha^j_1}$, so that for $(X_n, \nu_n) \in \mathcal{S}_{x_n} \times \mathcal{S}_{v_n}$, $|\delta_{\alpha_n}| \le d_{\alpha_n} < \infty$ for some bound $d_{\alpha_n} > 0$. Let $\Phi_{\alpha^j_n} = \hat{\theta}_{\alpha^j_n} - \theta^*_{\alpha^j_n}$, and consider the approximation $\hat{\alpha}_n$ as given in Theorem 1. The control law $u = \hat{\alpha}_n + \alpha^s_n$ yields $\dot{z}_n = \phi^c_n + \psi^c_n(\hat{\alpha}_n + \alpha^s_n) - \dot{\hat{\alpha}}_{n-1} - \dot{\alpha}^s_{n-1} + \psi^c_n(\alpha^*_n - \alpha^*_n) = -c_n z_n + \psi^c_n\left(\sum_{j=1}^R \rho_j(v) \Phi^\top_{\alpha^j_n} \zeta_{\alpha^j_n} - \delta_{\alpha_n}\right) + \psi^c_n \alpha^s_n$. Choose the Lyapunov function candidate $V = V_{n-1} + \frac{1}{2\psi^c_n} z_n^2 + \frac{1}{2}\sum_{j=1}^R \frac{\Phi^\top_{\alpha^j_n} \Phi_{\alpha^j_n}}{\gamma_{\alpha^j_n}}$ and examine its derivative, $\dot{V} = \dot{V}_{n-1} - \frac{c_n z_n^2}{\psi^c_n} + z_n \sum_{j=1}^R \rho_j(v) \Phi^\top_{\alpha^j_n} \zeta_{\alpha^j_n} - z_n \delta_{\alpha_n} + z_n \alpha^s_n - \frac{1}{2} z_n^2 \frac{\dot{\psi}^c_n}{\psi^{c2}_n} + \sum_{j=1}^R \frac{\Phi^\top_{\alpha^j_n} \dot{\Phi}_{\alpha^j_n}}{\gamma_{\alpha^j_n}}$. One can show inductively that $\dot{V}_{n-1} \le -\sum_{i=1}^{n-1} \frac{\bar{c}_i z_i^2}{\bar{\psi}^c_i} - \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^R \sigma_{\alpha^j_i} \frac{|\Phi_{\alpha^j_i}|^2}{\gamma_{\alpha^j_i}} + z_{n-1} z_n + \sum_{i=1}^{n-1} \frac{d^2_{\alpha_i}}{4k_i} + \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^R \sigma_{\alpha^j_i} \frac{|\theta^*_{\alpha^j_i}|^2}{\gamma_{\alpha^j_i}}$ with constants $\bar{c}_i = c_i - \frac{\psi_{i_d}}{2\underline{\psi}^c_i} > 0$, $i = 1, \ldots, n$. The choice of adaptation laws for $\theta_{\alpha^j_n}$ and of $\alpha^s_n$ in Theorem 1, together with the observations that $-\frac{\sigma_{\alpha^j_n}}{\gamma_{\alpha^j_n}} \Phi^\top_{\alpha^j_n} \hat{\theta}_{\alpha^j_n} \le -\frac{\sigma_{\alpha^j_n}}{\gamma_{\alpha^j_n}} \frac{|\Phi_{\alpha^j_n}|^2}{2} + \frac{\sigma_{\alpha^j_n}}{\gamma_{\alpha^j_n}} \frac{|\theta^*_{\alpha^j_n}|^2}{2}$, $-z_n \delta_{\alpha^j_n} \le k_n z_n^2 + \frac{d_{\alpha_n}}{4k_n}$, with $k_n > 0$ and $-\frac{z_n^2}{\psi^c_n}\left(c_n + \frac{\dot{\psi}^c_n}{2\psi^c_n}\right) \le -\frac{\bar{c}_n z_n^2}{\bar{\psi}^c_n}$ imply

$$\dot{V} \le -\sum_{i=1}^n \frac{\bar{c}_i z_i^2}{\bar{\psi}^c_i} - \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^R \sigma_{\alpha^j_i} \frac{|\Phi_{\alpha^j_i}|^2}{\gamma_{\alpha^j_i}} + W_d, \tag{5}$$

where $W_d$ contains the combined effects of representation errors and ideal parameter sizes, and is given by $W_d = \sum_{i=1}^n \frac{d^2_{\alpha_i}}{4k_i} + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^R \sigma_{\alpha^j_i} \frac{|\theta^*_{\alpha^j_i}|^2}{\gamma_{\alpha^j_i}}$. Note that if $\sum_{i=1}^n \frac{\bar{c}_i z_i^2}{\bar{\psi}^c_i} \ge W_d$ or $\frac{1}{2}\sum_{i=1}^n \sum_{j=1}^R \sigma_{\alpha^j_i} \frac{|\Phi_{\alpha^j_i}|^2}{\gamma_{\alpha^j_i}} \ge W_d$, then we have $\dot{V} \le 0$. Furthermore, letting $\underline{\psi}_m = \min_{1 \le i \le n}(\underline{\psi}^c_i)$, $\bar{\psi}_m = \max_{1 \le i \le n}(\bar{\psi}^c_i)$, and

[2] We omit intermediate steps for brevity.

defining $\bar{c}_0 = \min_{1\le i\le n}(\bar{c}_i)$, $\psi_m = \frac{\underline{\psi}_m}{\bar{\psi}_m}$ and $\sigma_0 = \min_{1\le i\le n, 1\le j\le R}\left(\sigma_{\alpha_i^j}\right)$ we have $-\sum_{i=1}^n \frac{\bar{c}_i z_i^2}{\psi_i^c} \le -\bar{c}_0 \sum_{i=1}^n \frac{z_i^2}{\psi_i^c} = -\bar{c}_0 \sum_{i=1}^n \frac{z_i^2}{\psi_i^c}\frac{\psi_i^c}{\bar{\psi}_i^c} \le -\bar{c}_0 \sum_{i=1}^n \frac{z_i^2}{\psi_i^c}\frac{\underline{\psi}_i^c}{\bar{\psi}_i^c} \le -\bar{c}_0\psi_m \sum_{i=1}^n \frac{z_i^2}{\psi_i^c}$ and $-\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^R \sigma_{\alpha_i^j}\frac{|\Phi_{\alpha_i^j}|^2}{\gamma_{\alpha_i^j}} \le -\sigma_0 \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^R \frac{|\Phi_{\alpha_i^j}|^2}{\gamma_{\alpha_i^j}}$. Then, letting $\beta_d = \min(2\bar{c}_0\psi_m, \sigma_0)$, we have that if

$$V = \frac{1}{2}\sum_{i=1}^n \frac{z_i^2}{\psi_i^c} + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^R \frac{|\Phi_{\alpha_i^j}|^2}{\gamma_{\alpha_i^j}} \ge V_0 \tag{6}$$

with $V_0 = \frac{W_d}{\beta_d}$, then $\dot{V} \le 0$ and all signals in the closed loop are bounded. Furthermore, we have $\dot{V} \le -\beta_d V + W_d$, which implies that $0 \le V(t) \le \frac{W_d}{\beta_d} + \left(V(0) - \frac{W_d}{\beta_d}\right)e^{-\beta_d t}$ so that both the transformed states and the parameter error vectors converge to a bounded set. Finally, we conclude from the upper bound on $V(t)$ that the state vector $X_n$ converges to the residual set (4). $\square$

**Remark 1**: The representation error bounds and the size of the ideal parameter vectors are assumed known, since they affect the size of the residual set to which the states converge. It is possible to augment the direct adaptive algorithm with "auto-tuning" capabilities (similar to [7]), which would relax the need for these bounds.

Furthermore, note that the stability result of Theorem 1 is semi-global, in the sense that it is valid within the compact sets $\mathcal{S}_{v_i}$ and $\mathcal{S}_{x_i}$, $i = 1, \ldots, n$, which can be made arbitrarily large. The stability result may be made global by adding a high gain bounding control term to the control law. Such a term may be particularly useful when, due to a complete lack of a priori knowledge, the control designer is unable to guarantee that the compact sets $\mathcal{S}_{x_i}$, $i = 1, \ldots, n$, are large enough so that the state will not exit them before the controller has time to bring the state inside $\mathcal{D}_d$; moreover, it may also happen that due to a poor design and poor system knowledge, $\mathcal{D}_d$ is not contained in $\mathcal{S}_{x_n}$. In this case, too, bounding control terms may be helpful until the design is refined and improved. However, using bounding control requires explicit knowledge of functional upper bounds of $|\psi_i^c(v, X_i)|$, and also of the lower bounds $\underline{\psi}_i^c$, $i = 1, \ldots, n$, whose knowledge we do not mandate in Theorem 1. Bounding terms may be added to the diffeomorphism in Theorem 1, but we do not present the analysis since it is similar to the one we present here and it is algebraically

tedious; we simply note, though, that the bounding terms have to be smooth (because they need to be differentiable), so they need to be defined in terms of smooth approximations to the sign, saturation and absolute value functions that are typically used in this approach.

**Remark 2**: If the bounds $\underline{\psi}_i^c$, $\bar{\psi}_i^c$ and $\psi_{i_d}^c$ are known, it becomes possible for the designer to directly set the constants $c_i$ in the control law. Notice that with knowledge of these bounds, the term $\underline{\psi}_m$ is also known, and we can pick constants $c_i$ such that $c_i > \frac{\psi_{i_d}^c}{2\underline{\psi}_i^c}$. Define the auxiliary functions $\eta_i = c_i z_i$. We may explicitly set the constant $c_i$ in $\alpha_i^*$ if we let $\eta_i$ be an input to the $i^{th}$ approximator structure, i.e., if we let $\alpha_i^*(X_i, \nu_i, \dot{X}_{r_i}, \eta_i) = \sum_{j=1}^R \rho_j(v)\theta_{\alpha_i^j}^{*\top} \zeta_{\alpha_i^j}(X_i, \nu_i, \dot{X}_{r_i}, \eta_i) + \delta_{\alpha_i}$. Then, the approximators used in the control procedure are given by $\hat{\alpha}_i(X_i, \nu_i, \dot{X}_{r_i}, \eta_i) = \sum_{j=1}^R \rho_j(v)\hat{\theta}_{\alpha_i^j}^\top \zeta_{\alpha_i^j}(X_i, \nu_i, \dot{X}_{r_i}, \eta_i)$ and the stability analysis can be carried out as expected.

## 2.2 Performance Analysis: $\mathcal{L}_\infty$ Bounds and Transient Design

The stability result of Theorem 1 is useful in that it indicates conditions to obtain a stable closed-loop behavior for a plant belonging to the class given by (1). However, it is not immediately clear how to choose the several design constants to improve the control performance. Here we concentrate on the tracking problem, and present design guidelines with respect to an $\mathcal{L}_\infty$ bound on the tracking error. We are interested in having $x_1$ track the reference model state $x_{r_1}$ of the reference model $\dot{x}_{r_i} = x_{r_{i+1}}$, $i = 1, 2, \ldots, n-1$, $\dot{x}_{r_n} = f_r(X_{r_n}, r)$ with bounded reference input $r(t) \in \mathbb{R}$. Now, we need to use the diffeomorphism $z_1 = x_1 - x_{r_1}$, $z_i = x_i - \hat{\alpha}_{i-1} - \alpha_{i-1}^s$, $i = 2, \ldots, n$ with $\alpha_1^*(x_1, v, \dot{x}_{r_1}) = \frac{1}{\psi_1^c}\left(-\phi_1^c - c_1 z_1 + x_{r_2}\right)$ and $\alpha_i^*(X_i, \nu_i, \dot{X}_{r_i}) = \frac{1}{\psi_i^c}\left(-\phi_i^c - c_i z_i + \dot{\hat{\alpha}}_i + \dot{\alpha}_i^s\right)$ for $i = 2, \ldots, n$. The stability proof needs to be modified accordingly, and it can be shown that the tracking error $|x_1 - x_{r_1}|$ converges to a neighborhood of size $\sqrt{\frac{2\underline{\psi}_m W_d}{\beta_d}}$.

From the upper bound on $V(t)$ we can write $V(t) \leq \frac{W_d}{\beta_d} + V(0)e^{-\beta_d t}$. From here, it follows that $\frac{1}{2}\sum_{i=1}^n \frac{z_i^2(t)}{\psi_i^c(t)} \leq \frac{W_d}{\beta_d} + \left(\frac{1}{2}\sum_{i=1}^n \frac{z_i^2(0)}{\psi_i^c(0)} + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^R \frac{|\Phi_{\alpha_i^j}(0)|^2}{\gamma_{\alpha_i^j}}\right) e^{-\beta_d t}$. The terms $z_i(0)$ depend on the design constants in a complex manner. For this reason, rather than trying to take them into account

in the design procedure, we follow the trajectory initialization approach taken in [6], which allows the designer to set $z_i(0) = 0$, $i = 1, \ldots, n$ by an appropriate choice of the reference model's initial conditions. In our case, in addition to the assumption that it is possible to set the initial conditions of the reference model, we will have to assume certain invertibility conditions on the approximators. In particular, since $z_1(0) = x_1(0) - x_{r_1}(0)$, for $z_1(0) = 0$ we need to set $x_{r_1}(0) = x_1(0)$.

For the $i^{th}$ transformed state $z_i$, $i = 2, \ldots, n$, $z_i(0) = x_i(0) - \hat{\alpha}_{i-1}(0) - \alpha^s_{i-1}(0)$. Notice that $\alpha^s_{i-1}(0) = \alpha^s_{i-1}(z_{i-1}(0), z_{i-2}(0))$, so that if $z_{i-1}(0) = 0$ and $z_{i-2}(0) = 0$ we have $\alpha^s_{i-1}(0) = 0$. In particular, notice that this holds for $i = 2$. In this case, to set $z_2(0) = 0$ we need to have $\hat{\alpha}_1(x_1(0), v(0), x_{r_2}(0)) = x_2(0)$. This equation can be solved analytically (or numerically) for $x_{r_2}(0)$ provided $\left.\frac{\partial \hat{\alpha}_1}{\partial x_{r_2}}\right|_{t=0} \neq 0$. This is not an unreasonable condition, since it depends on the choice of approximator structure the designer makes. The structure can be chosen so that it satisfies this condition. Granted this is the case, it clearly holds that $\alpha^s_2(0) = 0$, and the same procedure can be inductively carried out for $i = 3, \ldots, n$, with the choices $\hat{\alpha}_{i-1}(X_{i-1}(0), \nu_{i-1}(0), x_{r_i}(0)) = x_i(0)$.

This procedure yields the simpler bound $\sum_{i=1}^n z_i^2(t) \leq \frac{2\underline{\psi}_m W_d}{\beta_d} + \underline{\psi}_m \left( \sum_{i=1}^n \sum_{j=1}^R \frac{|\Phi_{\alpha_i^j}(0)|^2}{\gamma_{\alpha_i^j}} \right) e^{-\beta_d t}$. We would like to make this bound small, so that the transient excursion of the tracking error is small. Notice that we do not have direct control on the size of $\beta_d$, since this term depends on the unknown constants $c_i$, which appear in the ideal signals $\alpha_i^*$. Even though it is not necessary to be able to set $\beta_d$ to reduce the size of the bound, it is possible to do so if the bounds $\underline{\psi}_i^c$, $\bar{\psi}_i^c$ and $\psi_{i_d}^c$ are known.

At this point, it becomes more clear how to choose the constants to achieve a smaller bound. Recalling the expression of $W_d$, note that, first, one may want to have $\beta_d > 1$, so that $W_d$ is not made larger when divided by $\beta_d$, and so that the convergence is faster. This may be achieved by setting $c_i$ such that $2\bar{c}_i\psi_m > 1$ (if enough knowledge is available to do so) and $\sigma_{\alpha_i^j} > 1$. However, having large $\sigma_{\alpha_i^j}$ makes $W_d$ larger; this can be offset, however, by also choosing the ratio $\sigma_{\alpha_i^j}/\gamma_{\alpha_i^j} < 1$ or smaller. Finally, it is clear that making $k_i$ larger reduces the effects of the representation errors, and therefore makes $W_d$ smaller. Observe that there is enough design freedom to make $W_d$ small

and $\beta_d$ large independently of each other.

These simple guidelines may become very useful when performing a real control design. Moreover, notice that the bound on $\sum_{i=1}^{n} z_i^2(t)$ makes it possible to specify the compact sets of the approximators so that, even throughout the transient, it can be guaranteed that the states will remain within the compact sets without the need for a global bounding control term. This has been a recurrent shortcoming of many on-line function approximation based methods, and the explicit bound on the transient makes it possible to overcome it.

# 3 Conclusions

In this paper we have developed a direct adaptive control method for a class of uncertain nonlinear systems with a time-varying structure using a Lyapunov approach to construct the stability proofs. The systems we consider are composed of a finite number of "pieces," or dynamic subsystems, which are interpolated by functions that depend on a possibly exogenous scheduling variable. We assume that each piece is in strict feedback form, and show that the methods yield stability of all signals in the closed-loop, as well as convergence of the state vector to a residual set around the equilibrium, whose size can be set by the choice of several design parameters

We argue that the direct adaptive method presents several advantages over indirect methods in general, including the need for a smaller amount of information about the plant and a simpler design. Finally, we provide design guidelines based on $\mathcal{L}_\infty$ bounds on the transient and argue that this bound makes it possible to precisely determine how large the compact sets for the function approximators should be so that the states do not exit them.

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